COMBINATORICS OF RANK JUMPS IN SIMPLICIAL HYPERGEOMETRIC SYSTEMS

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Abstract. Let A be an integer $d \times n$ matrix, and assume that the convex hull $\operatorname{conv}(A)$ of its columns is a simplex of dimension $d-1$. It is known that the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay if and only if the rank of the GKZ hypergeometric system $H_A(\beta)$ equals the normalized volume of $\operatorname{conv}(A)$ for all complex parameters $\beta \in \mathbb{C}^d$ [Sai02]. Our refinement here shows that $H_A(\beta)$ has rank strictly larger than the volume of $\operatorname{conv}(A)$ if and only if β lies in the Zariski closure (in $\mathbb{C}^d$) of all $\mathbb{Z}^d$-graded degrees where the local cohomology $\bigoplus_{i<d} H^i_{\mathfrak{m}}(\mathbb{C}[\mathbb{N}A])$ is nonzero.

1. Introduction

Gelfand, Graev, Kapranov, and Zelevinsky [GGZ87, GZK89] defined certain linear systems of partial differential equations, now known as *A-hypergeometric* or *GKZ hypergeometric* systems $H_A(\beta)$, whose solutions generalize the classical hypergeometric series. These holonomic systems are constructed from discrete input consisting of an integer $d \times n$ matrix A, along with continuous input consisting of a complex vector $\beta \in \mathbb{C}^d$. The matrix A defines a semigroup ring $\mathbb{C}[\mathbb{N}A]$, and the same authors have shown that the dimension $\operatorname{rank}(H_A(\beta))$ of the space of analytic solutions of $H_A(\beta)$ is independent of β when $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay [GZK89].

Meanwhile, Adolphson showed that even when $\mathbb{C}[\mathbb{N}A]$ is not Cohen–Macaulay, the rank of $H_A(\beta)$ is independent of β, as long as β is generic in a certain precise sense [Ado94]. After Sturmfels and Takayama showed that the rank can actually go up for non-generic parameters β [ST98], Cattani, D'Andrea and Dickenstein [CDD99] showed that if $\operatorname{conv}(\mathbb{N}A)$ is a segment, then in fact the rank does jump whenever $\mathbb{C}[\mathbb{N}A]$ fails to be Cohen–Macaulay. This result was generalized by Saito [Sai02], who, using different methods, proved that there existed rank-jumping parameters for any non Cohen–Macaulay simplicial semigroup $\mathbb{C}[\mathbb{N}A]$.

In this note, we use the combinatorics of $\mathbb{Z}^d$-graded local cohomology to characterize the set of parameters β for which the rank goes up, in the simplicial case. Our premise, reviewed in Section 2, is the standard fact that a semigroup ring $\mathbb{C}[\mathbb{N}A]$ fails to be Cohen–Macaulay if and only if a local cohomology module $H^i_{\mathfrak{m}}(\mathbb{C}[\mathbb{N}A])$ is nonzero for some cohomological index i strictly less than than the dimension d of $\mathbb{C}[\mathbb{N}A]$. After gathering some facts about A-hypergeometric systems in Section 3, we show in Theorem 11 that the set of *rank jumping parameters* is the Zariski closure (in $\mathbb{C}^d$) of the set of all $\mathbb{Z}^d$-graded degrees where the local cohomology $\bigoplus_{i<d} H^i_{\mathfrak{m}}(\mathbb{C}[\mathbb{N}A])$ is nonzero.

The original role of this result was as evidence for our conjecture that it generalizes to arbitrary integer matrices A, regardless of whether or not the semigroup $\mathbb{N}A$ is simplicial. Although we do not know whether the methods of this note extend to the general case, our conjecture has since been proved using a different approach [MMW04].

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2. Computing local cohomology for semigroup rings

Throughout this note, let A be a $d \times n$ integer matrix whose first row has all entries equal to 1, and whose columns $a_1, \ldots, a_n$ generate $\mathbb{Z}^d$ as a group. Unless otherwise explicitly stated, we do not assume that the polytope $\mathrm{conv}(A)$ obtained by taking the convex hull (in $\mathbb{R}^d$) of the column vectors $a_1, \ldots, a_n$ is a simplex; in particular, we need no simplicial assumptions from here through Definition 10. The semigroup

$$\mathbb{N}A \quad = \quad \Big\{ \sum_{i=1}^n k_i a_i \mid k_1, \ldots, k_n \in \mathbb{N} \Big\}$$

has **semigroup ring** $R = \mathbb{C}[\mathbb{N}A] \cong \mathbb{C}[\partial_1, \ldots, \partial_n]/I_A$, where

$$I_A \quad = \quad \langle \partial^u - \partial^v \mid A \cdot u = A \cdot v \rangle$$

is the **toric ideal** of A. The ring R is naturally graded by $\mathbb{Z}^d$, with the i^{th} indeterminate having degree $\deg(\partial_i) = a_i$ equal to the i^{th} column of A. By a **face** of $\mathbb{N}A$ we mean a set of lattice points minimizing some linear functional on $\mathbb{N}A$. The terms **ray** and **facet** refer to faces of dimension 1 and $d-1$, respectively, where the dimension of a face equals the rank of the subgroup $\mathbb{Z}\tau \subseteq \mathbb{Z}^d$ it generates. It is convenient to identify a face τ of $\mathbb{N}A$ with the subset of $\{1, \ldots, n\}$ indexing the vectors a_i lying in τ.

We now recall some facts from [BH93, Chapter 6] about the local cohomology modules $H^i_{\mathfrak{m}}(R)$, where $\mathfrak{m} = \langle \partial_1, \ldots \partial_n \rangle$ is the graded maximal ideal of R. Since R is a semigroup ring, the local cohomology of R is the cohomology of the complex

$$0 \to R \to \bigoplus_{\text{rays } \tau} R_\tau \to \bigoplus_{\text{2-dim faces } \tau} R_\tau \to \cdots \to \bigoplus_{\text{facets } \tau} R_\tau \to R_{\mathfrak{m}} \to 0, \tag{1}$$

where R_τ is the localization of R by inverting the indeterminates ∂_i for $i \in \tau$. The differential is derived from the algebraic cochain complex of the polytope $\mathrm{conv}(A)$, once orientations on the faces of $\mathrm{conv}(A)$ have been chosen.

The above local cohomology can be computed multidegree by multidegree. Indeed, the localization R_τ is nonzero in graded degree $\beta \in \mathbb{Z}^d$ if and only if β lies in the subsemigroup $\mathbb{N}A + \mathbb{Z}\tau$ of $\mathbb{Z}^d$; in other words, $R_\tau = \mathbb{C}[\mathbb{N}A + \mathbb{Z}\tau]$. Therefore, the faces of $\mathbb{N}A$ contributing a nonzero vector space (of dimension 1) to the degree β piece of the complex (1) is

$$\nabla(\beta) \quad = \quad \{\text{faces } \tau \text{ of } \mathbb{N}A \mid \beta \in \mathbb{N}A + \mathbb{Z}\tau\}.$$

This set of faces is closed under going up, meaning that if $\tau \subset \sigma$ and $\tau \in \nabla(\beta)$, then also $\sigma \in \nabla(\beta)$. When we write cohomology groups $H^j(\nabla)$ for such a **polyhedral cocomplex**, what we mean formally is that

$$H^j(\nabla) \quad = \quad H^j(\mathrm{conv}(A), \mathrm{conv}(A) \smallsetminus \nabla; \mathbb{C})$$

is the cohomology with complex coefficients of the relative cochain complex of the complementary polyhedral subcomplex of $\mathrm{conv}(A)$.

Here is a standard result in combinatorial commutative algebra.

Theorem 1. *The local cohomology $H^j_{\mathfrak{m}}(R)_\beta$ of the semigroup ring R in $\mathbb{Z}^d$-graded degree β is isomorphic to $H^j(\nabla(\beta))$. In particular, R is Cohen–Macaulay if and only if $H^j(\nabla(\beta)) = 0$ for all degrees $\beta \in \mathbb{Z}^d$ and cohomological degerees $j = 0, \ldots, d-1$.*

Proof. Use the complex in (1) to compute local cohomology. □

Definition 2. [HM03] The **sector partition** is the partition of $\mathbb{Z}^d$ into equivalence classes for which $\beta \equiv \beta'$ if and only if $\nabla(\beta) = \nabla(\beta')$. For a cocomplex ∇, the (possibly empty) of degrees $\beta \in \mathbb{Z}^d$ satisfying $\nabla(\beta) = \nabla$ is a **sector**.

Since the local cohomology of R in degree β only depends on $\nabla(\beta)$, it is constant on every sector.

Definition 3. A degree $\beta \in \mathbb{Z}^d$ such that $H^j(\nabla(\beta)) \neq 0$ for some $0 \leq j \leq d-1$ is called an **exceptional degree** of A. The Zariski closure of the set $E(A)$ of exceptional degrees of A is called the **slab arrangement** $\overline{E}(A)$ of A. Given an irreducible component of the slab arrangement, the set of exceptional degrees lying inside that component and in no other components is called a **slab**.

Proposition 4. *The slab arrangement is a union of affine translates of linear subspaces $\mathbb{C}\tau$ generated by faces τ of $\mathbb{N}A$, thought of as subsets of $\mathbb{C}^d$.*

Proof. The Matlis dual of each local cohomology module is finitely generated, and therefore has a filtration whose successive quotients are $\mathbb{Z}^d$-graded shifts of quotients of R by prime monomial ideals. Each successive quotient is therefore a $\mathbb{Z}^d$-graded shift of a semigroup ring $\mathbb{C}[\tau]$ for some face τ of $\mathbb{N}A$. The Matlis dual of local cohomology is thus supported on a set of degrees satisfying the conclusion of the proposition. The exceptional degrees are the negatives of the support degrees of the Matlis dual. □

3. Local cohomology and A-hypergeometric systems

Denote by D_n the **Weyl algebra**, by which we mean the ring of linear partial differential operators with polynomial coefficients in n variables. That is, D_n is the free associative algebra $\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ modulo the relations $x_i x_j - x_j x_i$, $\partial_i \partial_j - \partial_j \partial_i$ and $\partial_j x_i - x_i \partial_j - \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Definition 5. Given $A = (a_{ij})$ as before and $\beta \in \mathbb{C}^d$, the A**-hypergeometric system** with **parameter** β is the left ideal in the Weyl algebra D_n generated by

$$I_A \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \quad \text{for } i = 1, \ldots, d.$$

The A**-hypergeometric module** with **parameter** β is $M_A(\beta) = D_n / H_A(\beta)$.

The following result relates our way of computing local cohomology of semigroup rings to A-hypergeometric systems.

Theorem 6. *Stratify $\mathbb{Z}^d$ so that β lies in the same stratum as β' iff the D-modules $M_A(\beta)$ and $M_A(\beta')$ are isomorphic. This stratification refines the sector partition, meaning that $M_A(\beta) \cong M_A(\beta')$ implies $\nabla(\beta) = \nabla(\beta')$.*

To prove this result, we recall Saito's combinatorial results on isomorphisms of hypergeometric D-modules.

Definition 7. Let $\beta \in \mathbb{C}^d$ and τ a face of the cone $\mathbb{N}A$. Let

$$E_\tau(\beta) \quad = \quad \{\lambda \in \mathbb{C}\tau \mid \beta \in \lambda + \mathbb{N}A + \mathbb{Z}\tau\} / \mathbb{Z}\tau$$

be the set of vectors $\lambda \in \mathbb{C}\tau$, up to translation by $\mathbb{Z}\tau$, for which $\beta - \lambda$ lies in the localization of $\mathbb{N}A$ along τ.

Theorem 8. [Sai02] *The D-modules $M_A(\beta)$ and $M_A(\beta')$ are isomorphic for two parameters β and β' in $\mathbb{C}^d$ if and only if $E_\tau(\beta) = E_\tau(\beta')$ for all faces τ of $\mathbb{N}A$.*

Proof of Theorem 6. Given a vector $\beta \in \mathbb{Z}^d$, we have

$$\nabla(\beta) \quad = \quad \{\text{faces } \tau \text{ of } \mathbb{N}A \mid 0 \in E_\tau(\beta)\} \tag{2}$$

by definition. Therefore, for any pair of parameters $\beta, \beta' \in \mathbb{Z}^d$ such that $M_A(\beta)$ is isomorphic to $M_A(\beta')$, we conclude that $\nabla(\beta) = \nabla(\beta')$ by Theorem 8. □

Remark 9. In general, the refinement in Theorem 6 is proper.

4. Rank jumps in the simplex case

Definition 10. A parameter vector $\beta \in \mathbb{C}^d$ is an **rank-jumping parameter** of A if $\operatorname{rank}(H_A(\beta)) > \operatorname{vol}(A)$, where $\operatorname{vol}(A)$ is the normalized volume of the polytope $\operatorname{conv}(A)$. The set of rank-jumping parameters of A is called the **exceptional set** of A, and denoted $\mathcal{E}(A)$.

Theorem 11. *Fix a $d \times n$ integer matrix A. If $\operatorname{conv}(A)$ is a $(d-1)$-simplex, then the exceptional set $\mathcal{E}(A)$ equals the Zariski closure $\overline{E}(A)$ of the set of exceptional degrees.*

Remark 12. Computational evidence (using the computer algebra systems Macaulay 2 [GS], Singular [GPS01], and CoCoA [CoC]) as well as heuristic arguments led us to conjecture the statement of Theorem 11. In fact, the evidence suggested that Theorem 11 generalizes to the case where A is an arbitrary integer matrix. This has since been shown in a subsequent paper [MMW04] via general geometric and homological methods.

Before getting to the proof, we need four preliminary results. The first two do not invoke the hypothesis that $\operatorname{conv}(A)$ is a simplex.

Lemma 13. *Suppose that ρ is a face of $\mathbb{N}A$, and $\alpha \in \rho$ is a vector not lying on any proper face of ρ. If $\beta \in \mathbb{Z}^d$, the only localizations $\mathbb{N}A + \mathbb{Z}\mu$ capable of containing $\beta - m\alpha$ for all large (positive) integers m are those for faces μ containing ρ. In other words,*

$$\mu \in \nabla(\beta - m\alpha) \text{ for all } m \gg 0 \quad \Rightarrow \quad \mu \supseteq \rho.$$

Proof. If μ does not contain ρ, then choose a linear functional that is zero along μ but positive on α. This linear functional remains negative on $\beta - m\alpha + \gamma$ for all $m \gg 0$ and $\gamma \in \mu$, so that $\beta - m\alpha \notin \mathbb{N}A + \mathbb{Z}\mu$. □

Lemma 14. *Fix $\beta \in \mathbb{Z}^d$. Suppose that ρ is maximal among faces of $\mathbb{N}A$ not in $\nabla(\beta)$, but that ρ is neither $\mathbb{N}A$ nor a facet of $\mathbb{N}A$. If $\alpha \in \rho$ is a vector not lying on any proper face of ρ, then $\beta - m\alpha$ is an exceptional degree for all large integers m.*

Proof. Suppose that μ contains ρ. Since $m\alpha \in \mathbb{Z}\mu$ for all integers m, we find that $\beta - m\alpha \in \mathbb{N}A + \mathbb{Z}\mu$ for all integers m if and only if $\beta \in \mathbb{N}A + \mathbb{Z}\mu$. By Lemma 13 we conclude that $\nabla(\beta - m\alpha)$ is, for $m \gg 0$, the cocomplex of all faces strictly containing ρ. The cohomology of such a cocomplex is the same as that of a sphere having dimension $1 + \dim(\rho)$: a copy of $\mathbb{C}$ in dimension $1 + \dim(\rho)$ and zero elsewhere. This cohomology is not in cohomological degree d by the codimension hypothesis on ρ. □

Lemma 15. *Suppose that ∇ is a cocomplex inside of a simplex of dimension e. If the cohomology $H^j(\nabla)$ is nonzero for some $j < e$, then there is a face ξ of codimension at least 2 inside the simplex such that $\xi \notin \nabla$ but $\mu \in \nabla$ for all other faces μ containing ξ.*

Proof. The equivalent dual statement is easier to visualize: If Δ is a simplicial complex inside of a simplex, and the reduced homology $\tilde{H}_j(\Delta)$ is nonzero in some homological degree $j \geq 0$, then there is a face ξ of dimension at least 1 in the simplex such that $\xi \notin \Delta$ but $\mu \in \Delta$ for every proper face μ of ξ. Equivalently, Δ has a minimal *non*face of dimension at least 1. This statement reduces easily to the case where all vertices of the simplex lie in Δ, and in that case one notes that *every* nonface has dimension at least 1, assuming Δ has at least two vertices. But Δ has reduced homology in dimension $j \geq 0$, so it must have at least two vertices and at least one nonface. $\square$

Remark 16. Lemma 15 fails immediately for polyhedral cocomplexes that are not simplicial. Two parallel edges of a square (plus the interior cell of the square) form a polyhedral cocomplex that has cohomology of dimension 1 in cohomological degree 1, but the only two maximal nonfaces are the remaining two edges, of codimension 1.

Theorem 17 ([Sai02])**.** *Suppose that the polytope* $\mathrm{conv}(A)$ *is a simplex. Then* β *is a rank-jumping parameter of* A *if and only if there exist faces* σ *and* τ *of* $\mathbb{N}A$*, and an element* $\lambda \in \mathbb{C}\sigma \cap \mathbb{C}\tau$*, such that*

$$\lambda \in E_\sigma(\beta) \cap E_\tau(\beta) \quad \text{but} \quad \lambda \notin E_{\sigma\cap\tau}(\beta).$$

Proof of Theorem 11. We begin by showing that the exceptional set is contained in the slab arrangement. Let $\beta \in \mathbb{C}^d$ be a rank-jumping parameter of A, and pick σ, τ and λ as in Theorem 17. For any $\alpha \in \mathbb{C}\sigma \cap \mathbb{C}\tau$, the sum $\beta + \alpha$ is a rank-jumping parameter, as can be seen by replacing λ with $\lambda + \alpha$ in Theorem 17 and noting that

$$\lambda + \alpha \in E_\tau(\beta + \alpha) \iff \lambda \in E_\tau(\beta). \tag{3}$$

Therefore we may (and do) assume that $\beta \in \mathbb{Z}^d$ and $\lambda = 0$.

Recall (2), which in particular implies that the set of faces μ satisfying $0 \in E_\mu(\beta)$ forms a cocomplex. This allows us to enlarge σ and τ so that $\sigma \cap \tau$ is maximal among faces of $\mathbb{N}A$ outside $\nabla(\beta)$, while still satisfying Theorem 17. Taking $\rho = \sigma \cap \tau$ in Lemma 14, we find that $\beta - m\alpha$ is an exceptional degree for all $m \gg 0$ and all choices of α interior to ρ. The slab arrangement $\overline{E}(A)$ therefore contains $\beta + \mathbb{C}\rho$, and hence β.

Now suppose by Proposition 4 that $\beta + \mathbb{C}\rho$ is an irreducible component of the slab arrangement $\overline{E}(A)$, where $\beta \in \mathbb{Z}^d$ and ρ is a face of $\mathbb{N}A$. We wish to show that $\beta + \mathbb{C}\rho$ consists of rank-jumping parameters. In fact, we shall produce σ, τ, and λ as in Theorem 17 satisfying $\rho = \sigma \cap \tau$, although we might harmlessly shift β by some vector in $\mathbb{Z}^d \cap \mathbb{C}\rho$ first.

The component $\beta + \mathbb{C}\rho$ is the closure of some slab parallel to ρ, which must (perhaps after shifting β by an element in $\mathbb{Z}^d \cap \mathbb{C}\rho$) contain $\beta - m\alpha$ for an integer point α interior to ρ and all $m \gg 0$. Replace β by $\beta - m\alpha$ for some fixed large choice of m. Lemma 13 implies that the cocomplex $\nabla(\beta)$ is contained in the simplex consisting of all faces of $\mathbb{N}A$ containing ρ. If ρ has dimension $d - e - 1$, then this simplex satisfies the hypotheses of Lemma 15. Therefore we can find a face ξ containing ρ and of dimension at most $d-2$, such that ξ is a maximal nonface of $\nabla(\beta)$. Applying Lemma 14 to ξ instead of ρ, we find that the component $\beta + \mathbb{C}\xi$ contains $\beta + \mathbb{C}\rho$, and still lies inside $\overline{E}(A)$. From this we conclude that $\rho = \xi$, because $\beta + \mathbb{C}\rho$ is an irreducible component of $\overline{E}(A)$.

In summary, given that $\beta + \mathbb{C}\rho$ is an irreducible component of the slab arrangement $\overline{E}(A)$, we have moved β by an element in $\mathbb{Z}^d \cap \mathbb{C}\rho$ so that

$$\rho \notin \nabla(\beta), \quad \text{but} \quad \mu \in \nabla(\beta) \text{ for all faces } \mu \text{ strictly containing } \rho.$$

Moreover, we have shown that $\dim(\rho) \leq d - 2$. Therefore we can pick two faces σ and τ strictly containing ρ and satisfying $\rho = \sigma \cap \tau$. For each $\lambda \in \mathbb{C}\rho$, we find that $\beta + \lambda$ is a

rank-jumping parameter by substituting $\lambda = 0$ and $\alpha = \lambda$ in (3), then using (2), and finally applying Theorem 17. $\square$

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